Quantum algorithms

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(16-4-2009)

Deutsch-Jozsa

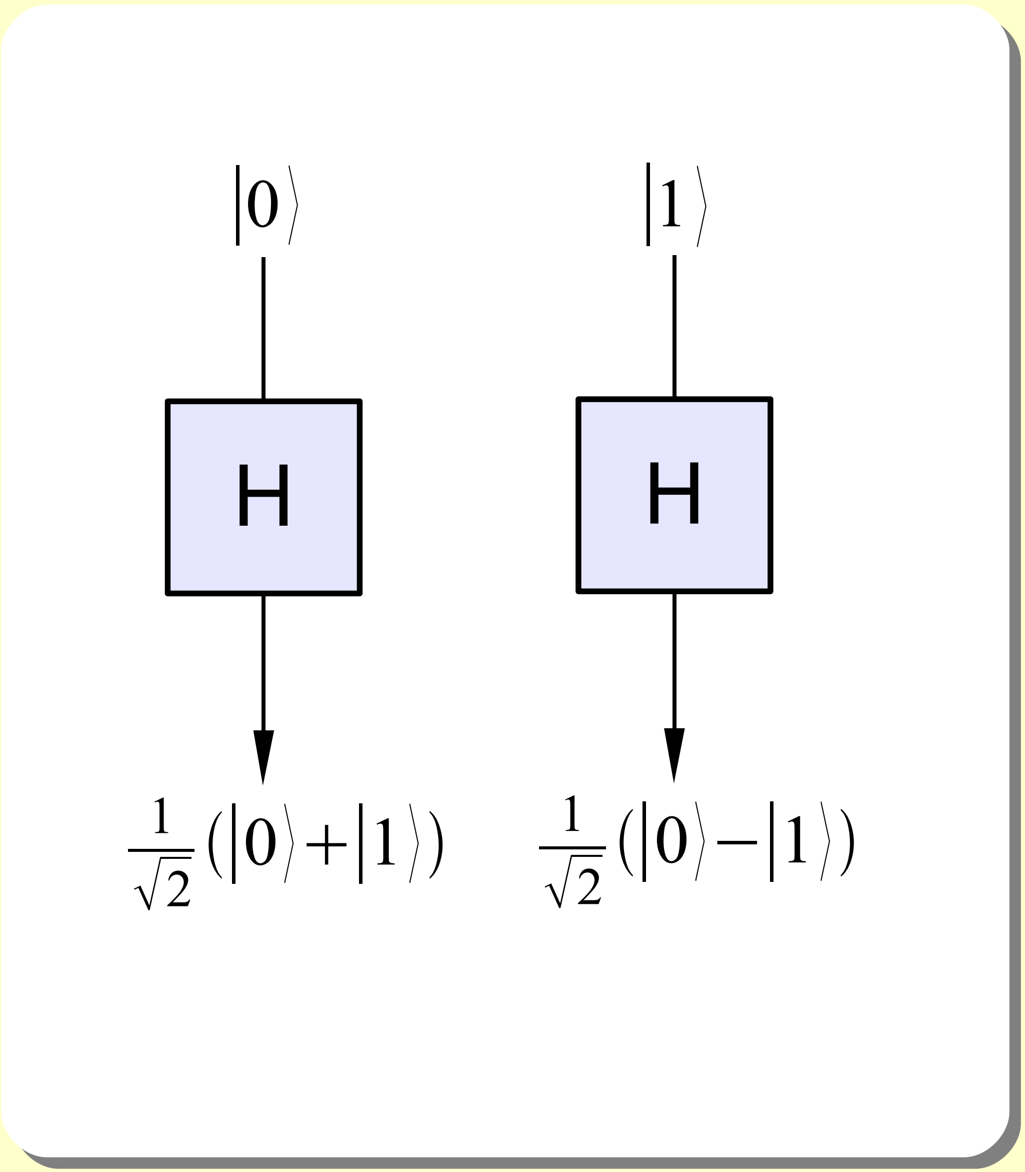
Deutsch-Jozsa

- The problem: we have a function f(x) that is either constant, or balanced

$$\begin{cases} f(x) = 1 \\ f(x) = 0 \\ \left| f^{-1}(0) \right| = \left| f^{-1}(1) \right| \end{cases}$$

- Classically we need 2^{N-1} evaluations.
- Probabilistically much less.
- Quantum mechanically?

Hadamard gate



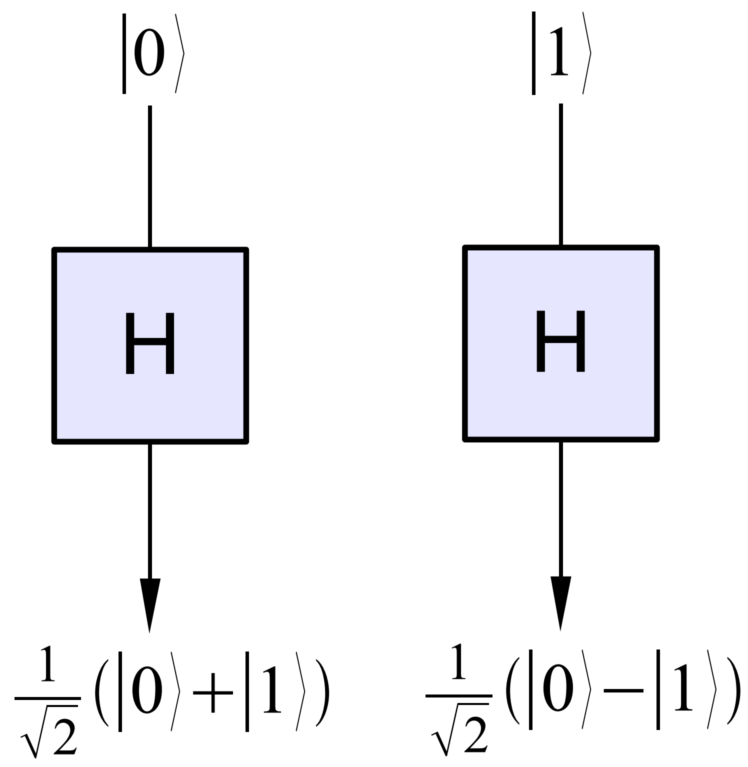
- The Hadamard gate transforms a qubit into a coherent superposition.
- It can be used to create a superposition of all possible values of N bits

$$H^{\otimes N}|0\ldots0\rangle = \sum_{x=0}^{2^N-1} |x\rangle$$

- But it also implements a Fourier transform

$$H|x\rangle = \sum_{y} (-1)^{x \oplus y} |y\rangle$$

Hadamard gate as FFT



- To see that it is a FFT for a 2-dimensional space,

$$|\psi\rangle = f(0)|0\rangle + f(1)|1\rangle$$

the transformed is

$$H|\psi\rangle = \sum_y \tilde{f}(y)|y\rangle$$

$$\tilde{f}(y) = \frac{1}{\sqrt{2}} \sum_x e^{i\pi(x+y)} f(x)$$

or

$$\tilde{f}(0) = \frac{1}{\sqrt{2}}[f(0)+f(1)]$$

$$\tilde{f}(1) = \frac{1}{\sqrt{2}}[f(0)-f(1)]$$

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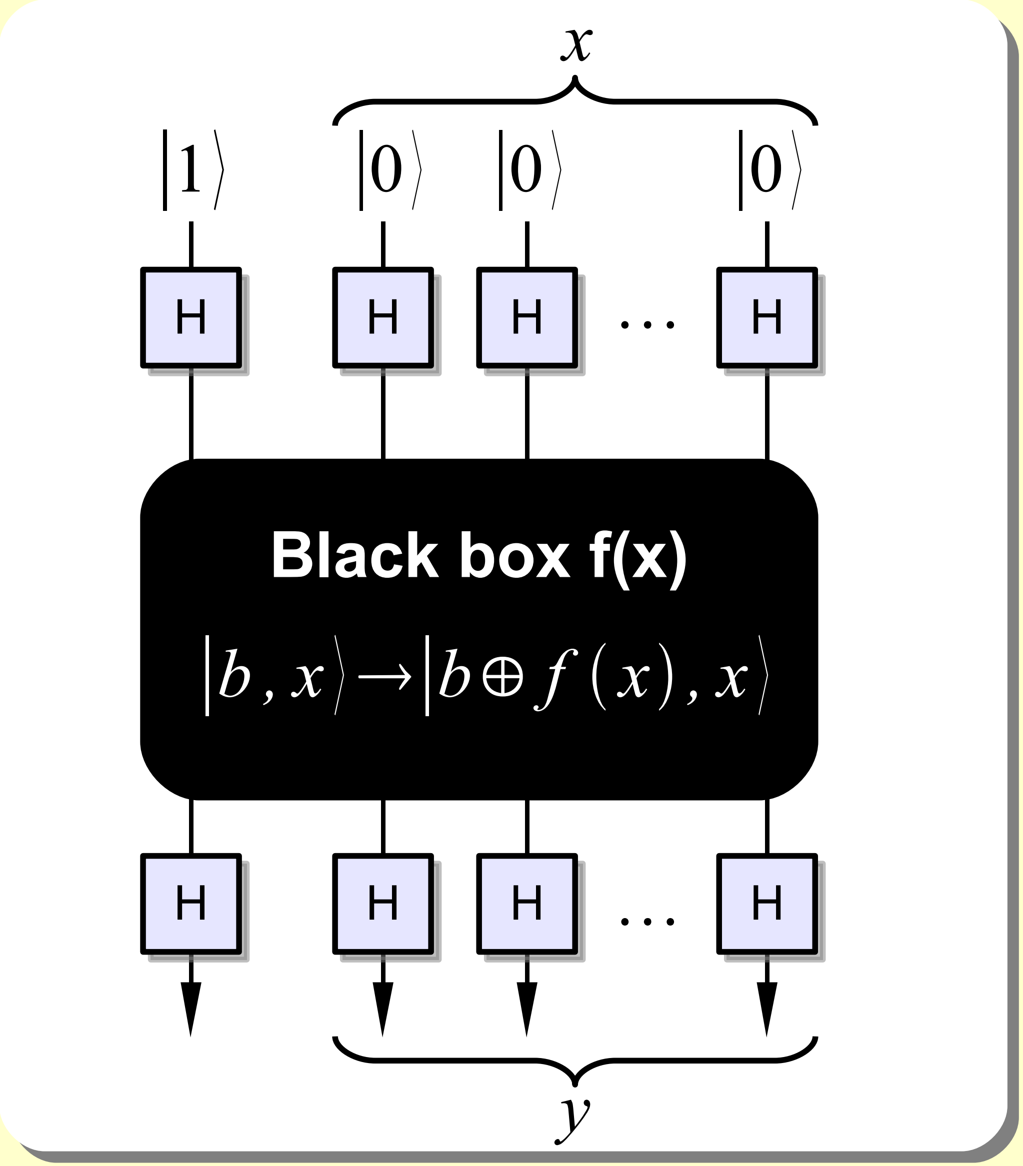


- We implement the function as a black box (needed for reversible computation!)
- After first two steps

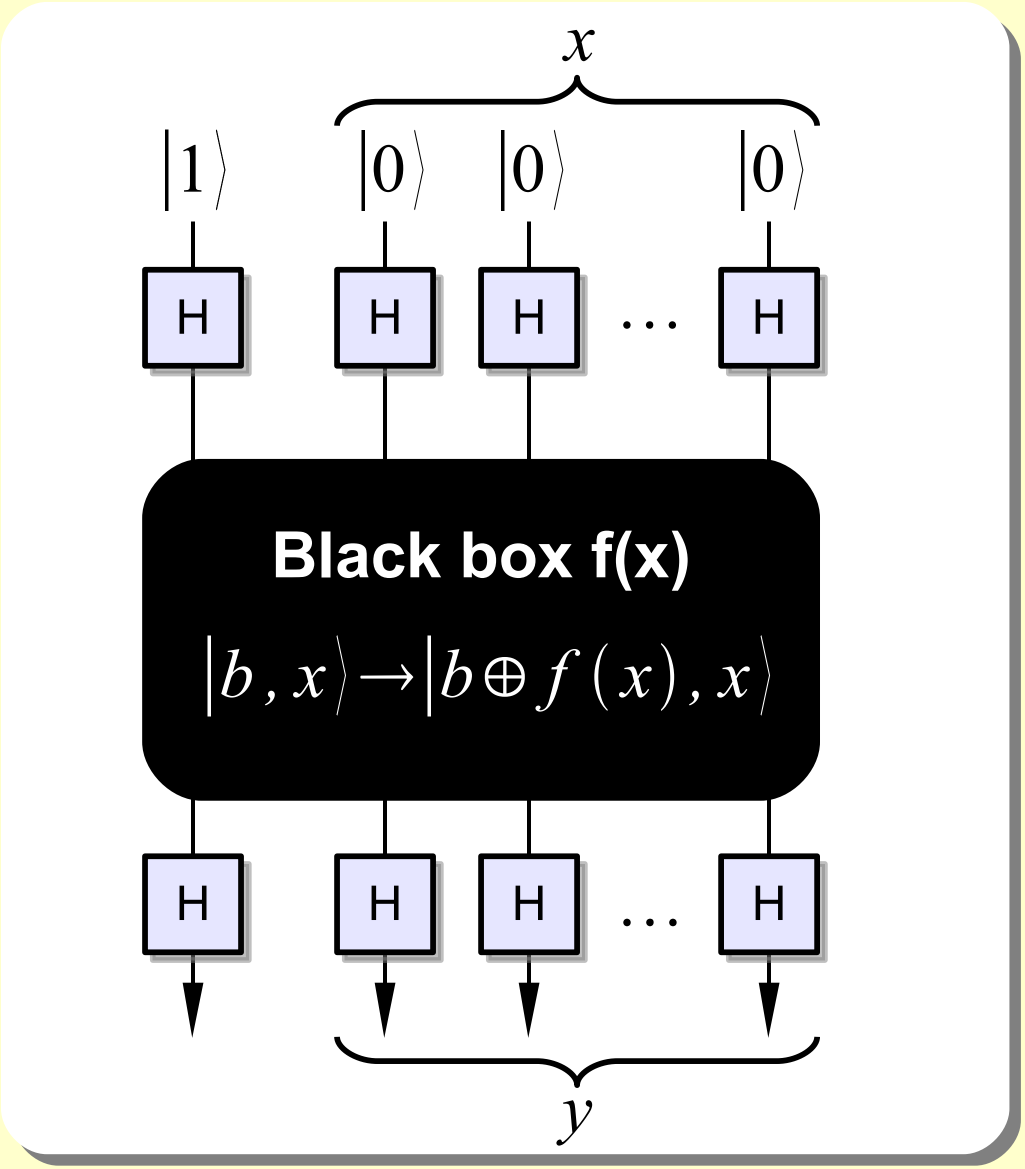
$$\sum_{x\in\mathbb{Z}_2^{\otimes N}} \left(|f(x)\rangle - |\bar{f}(x)\rangle\right)\otimes|x\rangle =$$

$$=\sum_x (-1)^{f(x)}\left(|0\rangle - |1\rangle\right)|x\rangle$$

$$=\sum_x (-1)^{f(x)}|-\rangle|x\rangle$$

- The first bit now factors out

Deutsch-Jozsa



- The second Hadamard implements the FFT

$$\sum_x (-1)^{f(x)} \sum_y (-1)^{x \cdot y} |y\rangle$$

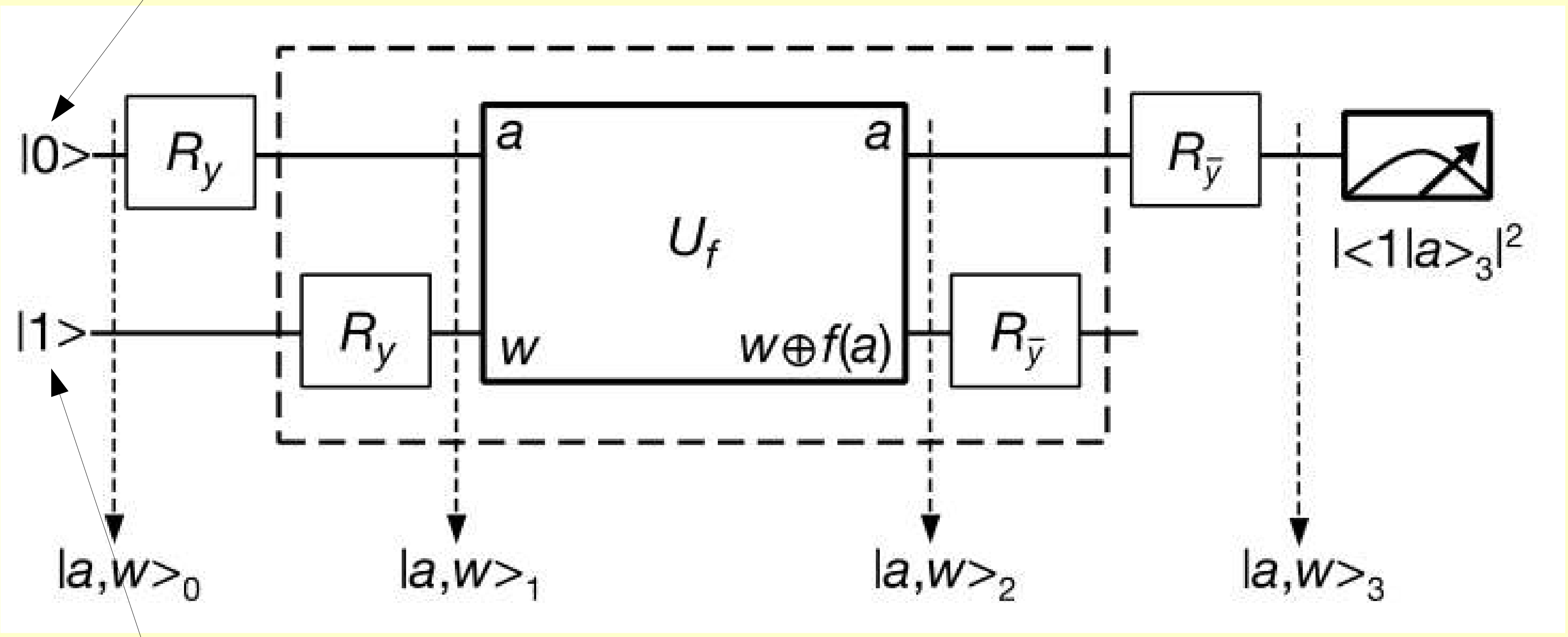
- We compute the amplitude of probability of getting y = 0

$$P(y=0) = \left| \sum_x (-1)^{f(x)} \right|^2$$

- So if f(x) is balanced always a nonzero qubit, otherwise all qubits become zero.

Deutsch algorithm with 1 ion

Atomic state qubit



Vibrational mode qubit

Implementation of oracle

	Constant functions		Balanced functions	
	Case 1	Case 2	Case 3	Case 4
$f(0)$	0	1	0	1
$f(1)$	0	1	1	0
$w \oplus f(a)$	ID	NOT	CNOT	Z-CNOT

	Logic	Laser pulses
f_1	$R_{\bar{y}_w} R_{y_w}$	No pulses
f_2	$R_{\bar{y}_w}$ SWAP^{-1} NOT$_a$ SWAP R_{y_w}	$R^+\left(\frac{\pi}{\sqrt{2}},0\right)R^+\left(\frac{2\pi}{\sqrt{2}},\varphi_{\mathrm{SWAP}}\right)R^+\left(\frac{\pi}{\sqrt{2}},0\right)$ $R(\frac{\pi}{2},0)R(\pi,\frac{\pi}{2})R(\frac{\pi}{2},\pi)$ $R^+\left(\frac{\pi}{\sqrt{2}},\pi\right)R^+\left(\frac{2\pi}{\sqrt{2}},\pi+\varphi_{\mathrm{SWAP}}\right)R^+\left(\frac{\pi}{\sqrt{2}},\pi\right)$
f_3	$R_{\bar{y}_w}$ CNOT R_{y_w}	$R^+\left(\frac{\pi}{\sqrt{2}},0\right)R^+(\pi,\frac{\pi}{2})R^+\left(\frac{\pi}{\sqrt{2}},0\right)R^+(\pi,\frac{\pi}{2})$
f_4	$R_{\bar{y}_w}$ Z-CNOT R_{y_w}	$R(\pi,0)R^+\left(\frac{\pi}{\sqrt{2}},0\right)R^+(\pi,\frac{\pi}{2})R^+\left(\frac{\pi}{\sqrt{2}},0\right)R^+(\pi,\frac{\pi}{2})R(\pi,0)$

Implementation of oracle

	Constant functions		Balanced functions	
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$f(0)$	0	1	0	1
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Laser pulses

No pulses

$$R^+\left(\frac{\pi}{\sqrt{2}},0\right)R^+\left(\frac{2\pi}{\sqrt{2}},\varphi_{SWAP}\right)R^+\left(\frac{\pi}{\sqrt{2}},0\right)$$

Carrier rotation →

$$R\left(\frac{\pi}{2},0\right)R\left(\pi,\frac{\pi}{2}\right)R\left(\frac{\pi}{2},\pi\right)$$

Blue sideband rotation →

$$R^+\left(\frac{\pi}{\sqrt{2}},\pi\right)R^+\left(\frac{2\pi}{\sqrt{2}},\pi+\varphi_{SWAP}\right)R^+\left(\frac{\pi}{\sqrt{2}},\pi\right)$$

$$R^+\left(\frac{\pi}{\sqrt{2}},0\right)R^+\left(\pi,\frac{\pi}{2}\right)R^+\left(\frac{\pi}{\sqrt{2}},0\right)R^+\left(\pi,\frac{\pi}{2}\right)$$

$$R(\pi,0)R^+\left(\frac{\pi}{\sqrt{2}},0\right)R^+\left(\pi,\frac{\pi}{2}\right)R^+\left(\frac{\pi}{\sqrt{2}},0\right)R^+\left(\pi,\frac{\pi}{2}\right)R(\pi,0)$$

Grover algorithm

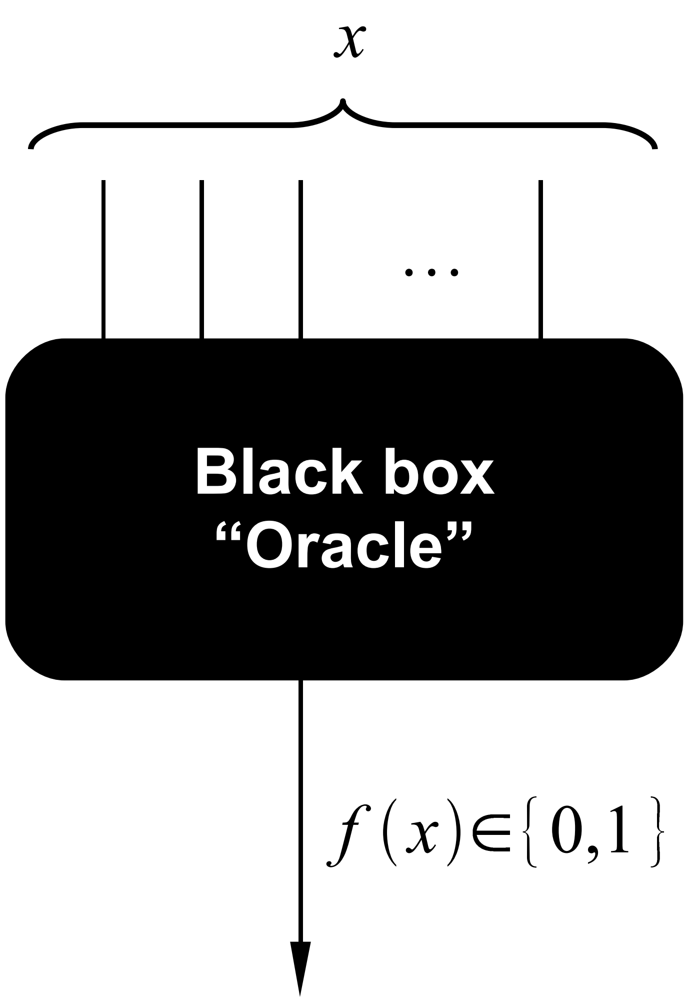
Grover algorithm

- More sophisticated algorithm, based also on an oracle.
- **Problem:** solve f(x)=1 for a binary function

$$f : \mathbb{Z}_2^{\otimes n} \rightarrow \{0,1\}$$

- The function "f" is our **database** and we are looking for one or more records.
- Classically, requires an exponentially large # of evaluations of f

$$O(N) = O(2^n)$$



The algorithm

- Prepare state

$$|\psi\rangle = \frac{1}{2^{n/2}} \sum_x |x\rangle$$

- Apply

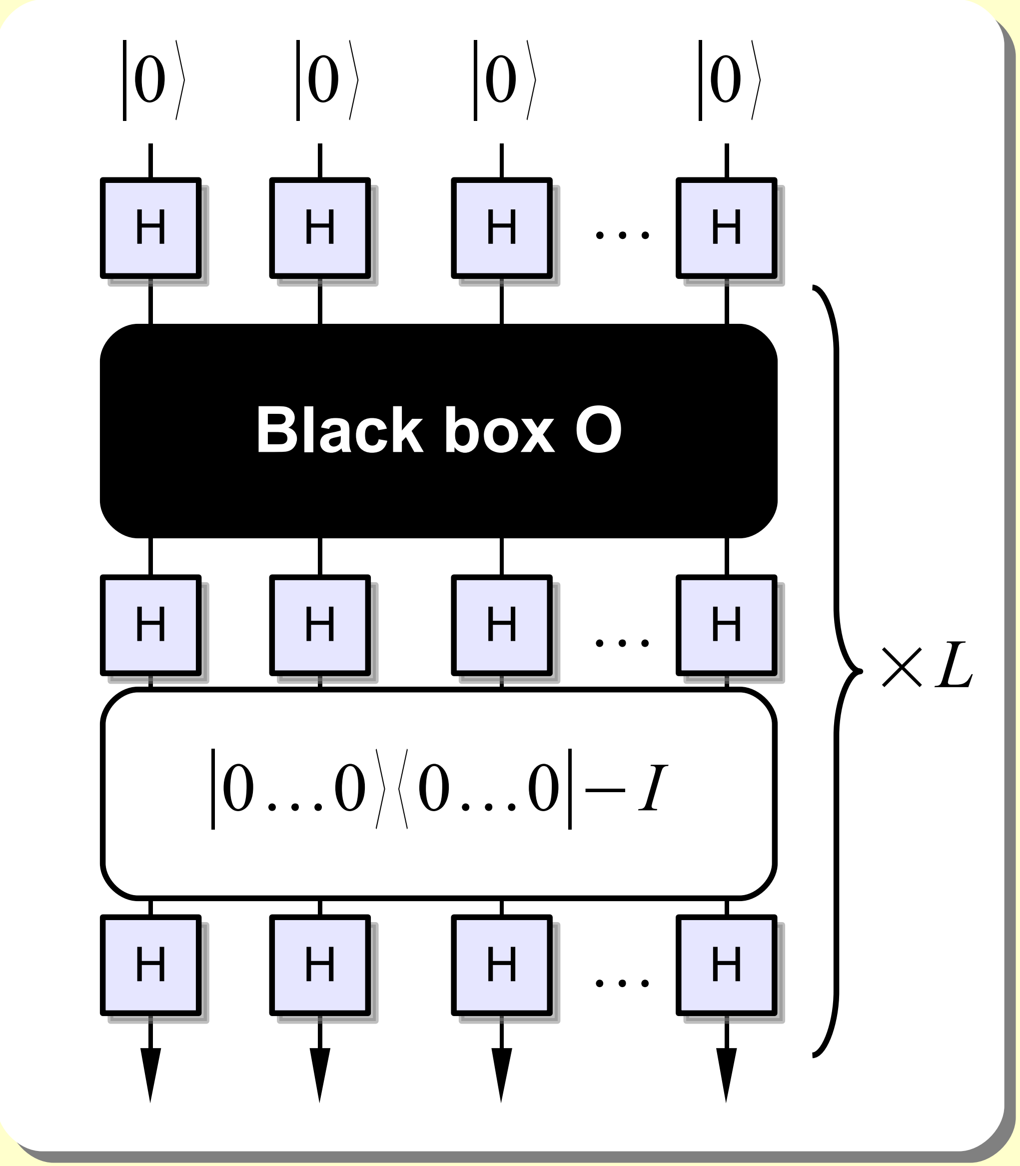
$$O = \sum_x (-1)^{f(x)} |x\rangle\langle x|$$

- Apply

$$R = H^{\otimes n} (|0\rangle\langle 0| - I) H^{\otimes n}$$

- Repeat from 2, L times

$$L = \sqrt{N/M}$$



Geometric model

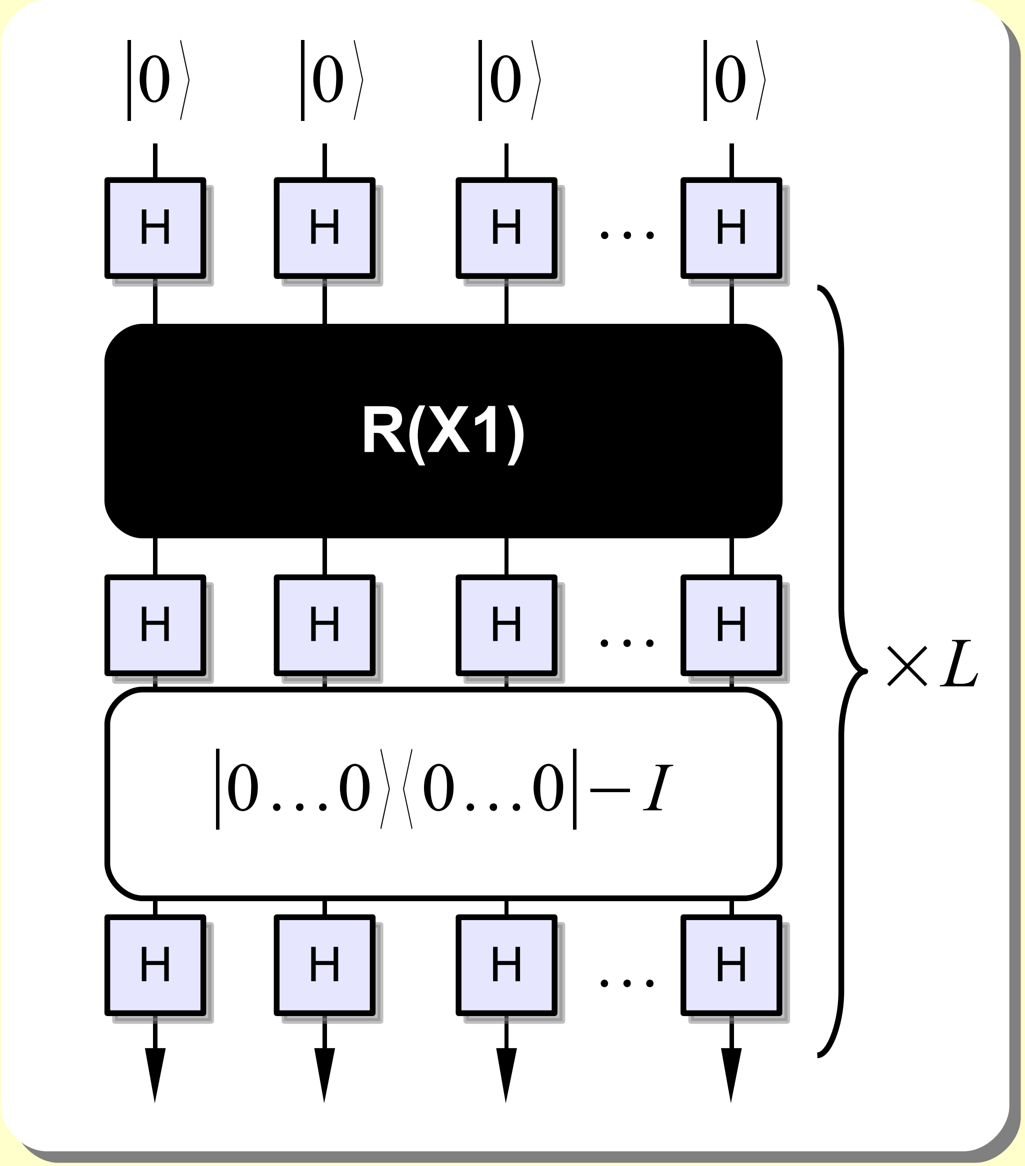
- Let us define

$$|X_1\rangle = \frac{1}{\sqrt{M}} \sum_x f(x)|x\rangle$$

$$|X_0\rangle = \frac{1}{\sqrt{N-M}} \sum_x [1-f(x)]|x\rangle$$

- These are superpositions of **all solutions** with either f=0 or f=1.
- The black box implements a reflection along one

$$O = |X_0\rangle\langle X_0| - |X_1\rangle\langle X_1| + \ldots$$



Geometric model

- The second step is

$$R = H^{\otimes n}(|0\rangle\langle 0| - I)H^{\otimes n}$$

is also a reflection

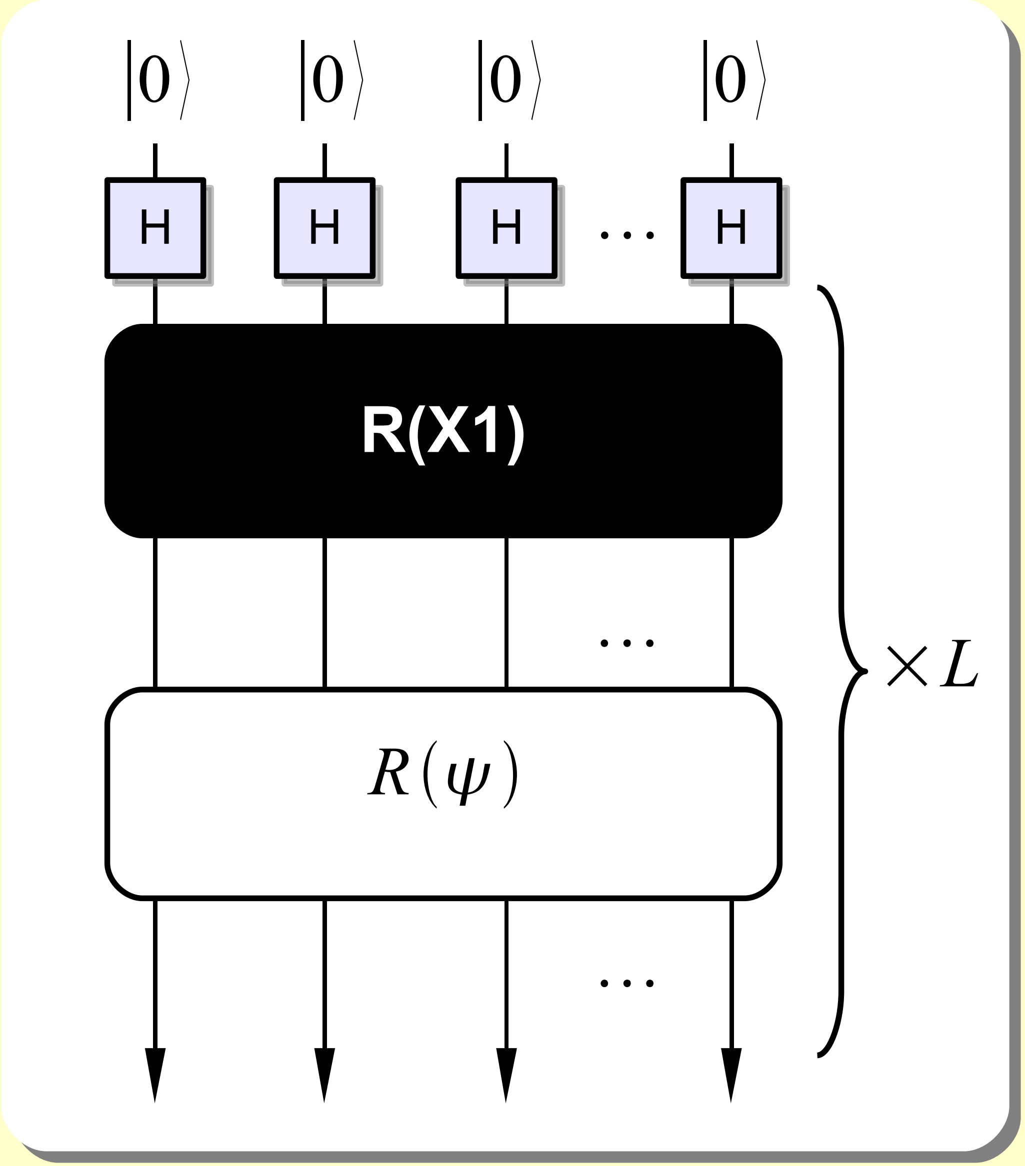
$$R = 2|\psi\rangle\langle\psi| - I$$

- Note that

$$|\psi\rangle = \sqrt{\frac{N-M}{N}}|X_0\rangle + \sqrt{\frac{M}{N}}|X_1\rangle$$

so that all steps keep us in the space {X0,X1}

$$(RO)^k|\psi\rangle = \cos(\theta_k)|X_1\rangle + \sin(\theta_k)|X_0\rangle$$



Geometric model

- Two reflections make up one rotation.
- Introducing the angle

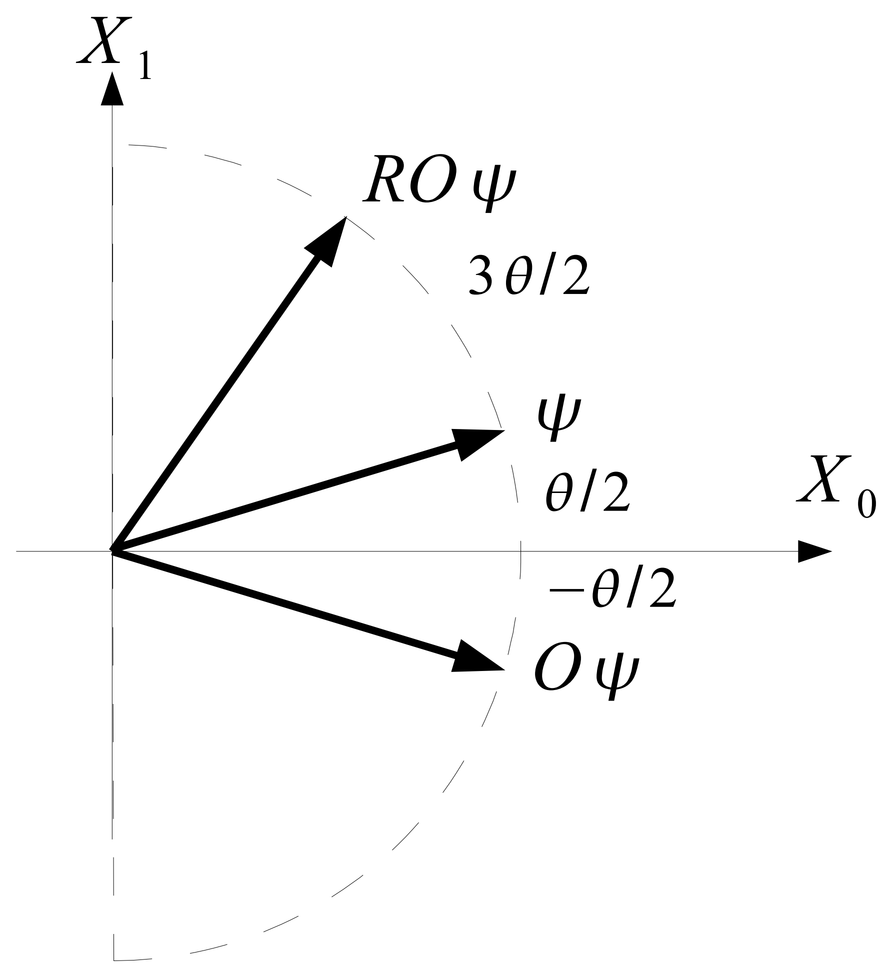
$$\psi = \cos(\theta/2) X_0 + \sin(\theta/2) X_1$$

we write

$$R = \begin{pmatrix} 2\cos(\theta/2)^2 - 1 & 2\sin(\theta/2)\cos(\theta/2) \\ 2\sin(\theta/2)\cos(\theta/2) & 2\sin(\theta/2)^2 - 1 \end{pmatrix}$$

or

$$RO = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$



Geometric model

Since we begin with

$$\psi = \cos(\theta/2)\, X_0 + \sin(\theta/2)\, X_1$$

after **k** steps we accumulate

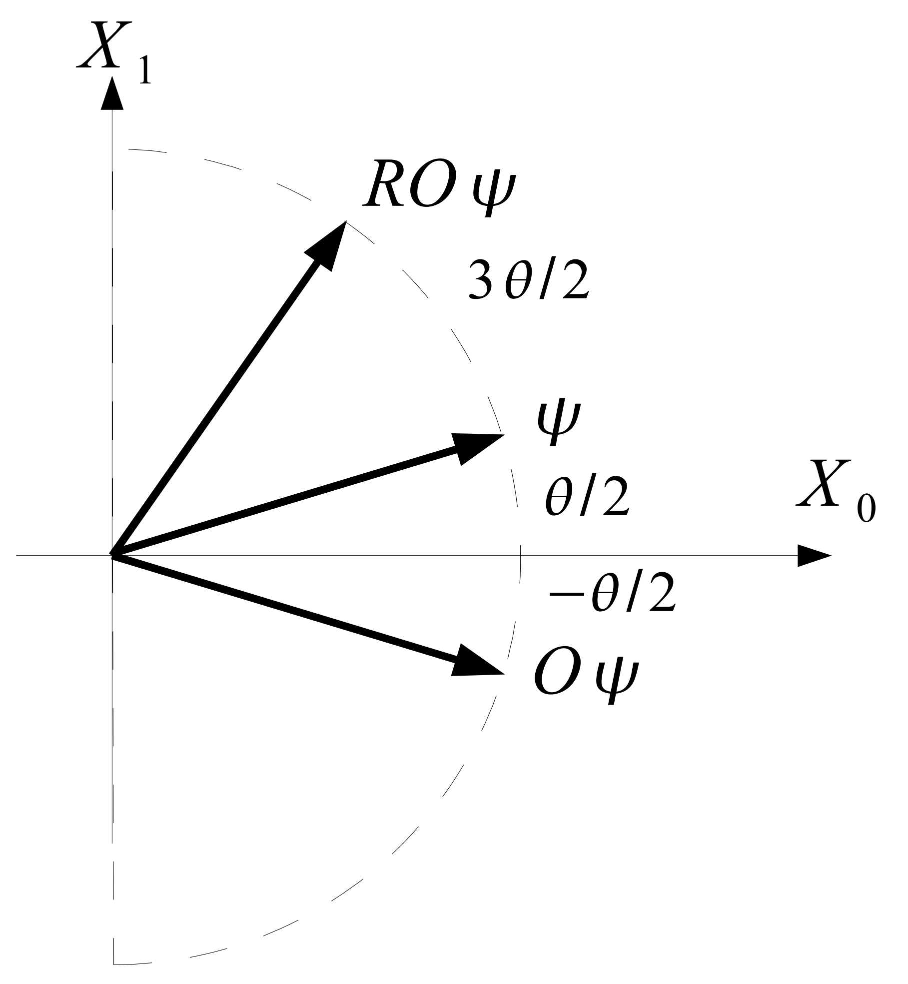
$$angle((RO)^k \psi, X_0) = \frac{2k+1}{2}\theta$$

The algorithm completes at

$$L = \left\lceil \frac{\pi}{2\theta} \right\rceil \simeq \frac{\pi}{4}\sqrt{\frac{N}{M}}$$

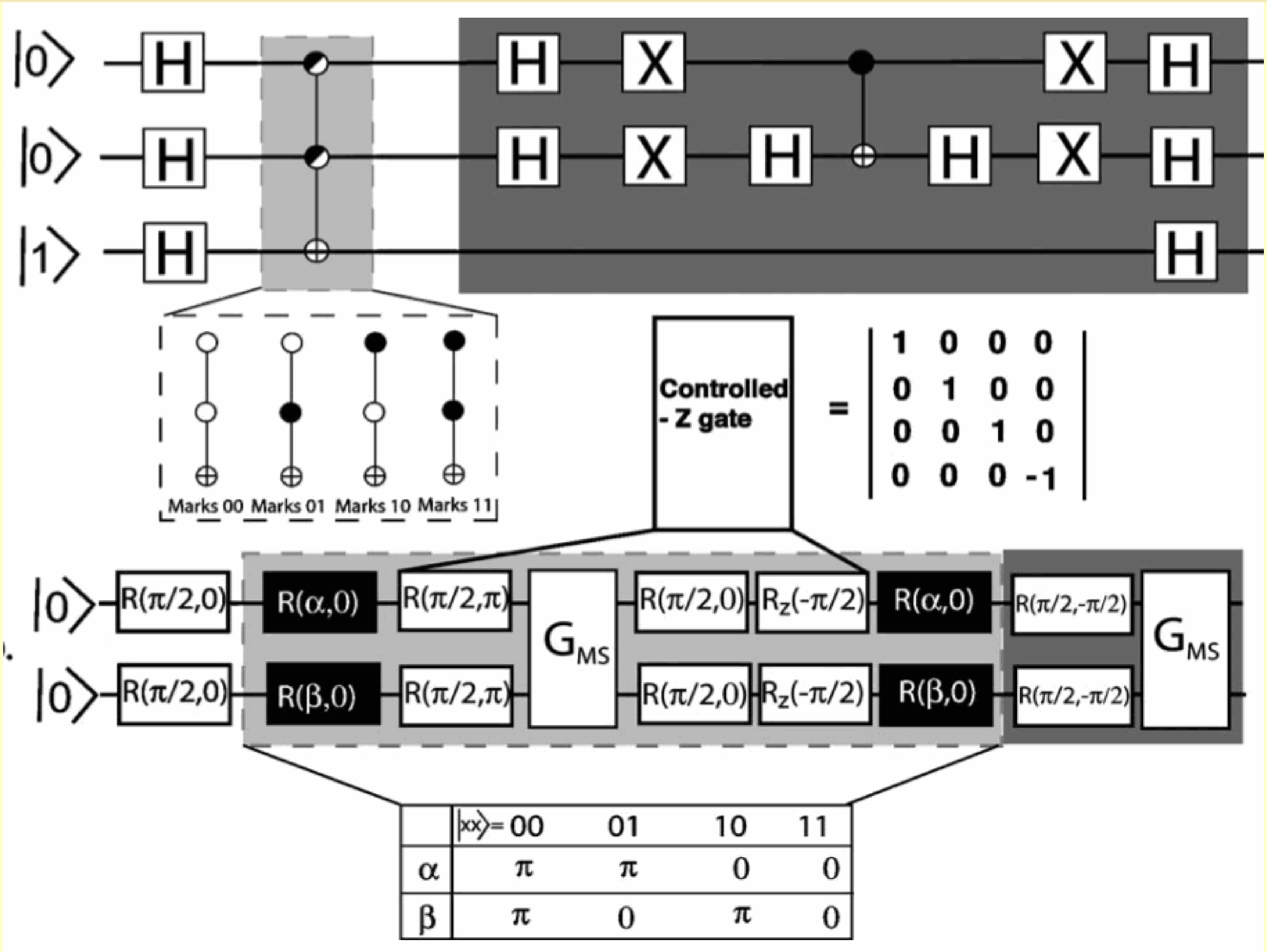
estimated for M << N using

$$\sin(\theta/2) \simeq \frac{\theta}{2} \simeq \sqrt{\frac{M}{N}}$$



Grover with ions

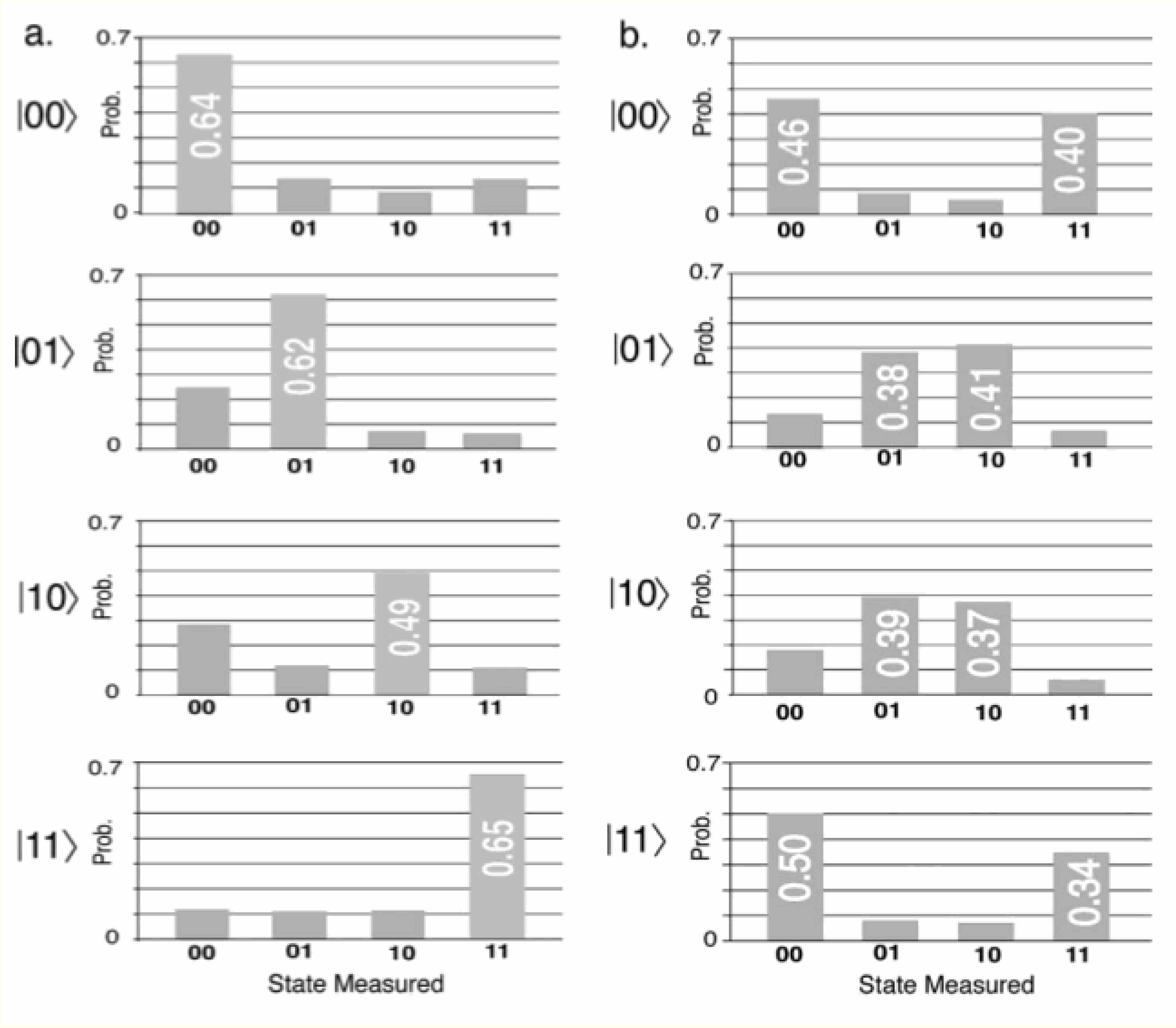
Grover with ions



K.-A. Brickman et al, Phys. Rev. A **72**, 050306 (2005)

Grover with ions

80% fidelity in the gate



K.-A. Brickman et al, Phys. Rev. A **72**, 050306 (2005)