Perturbation theory

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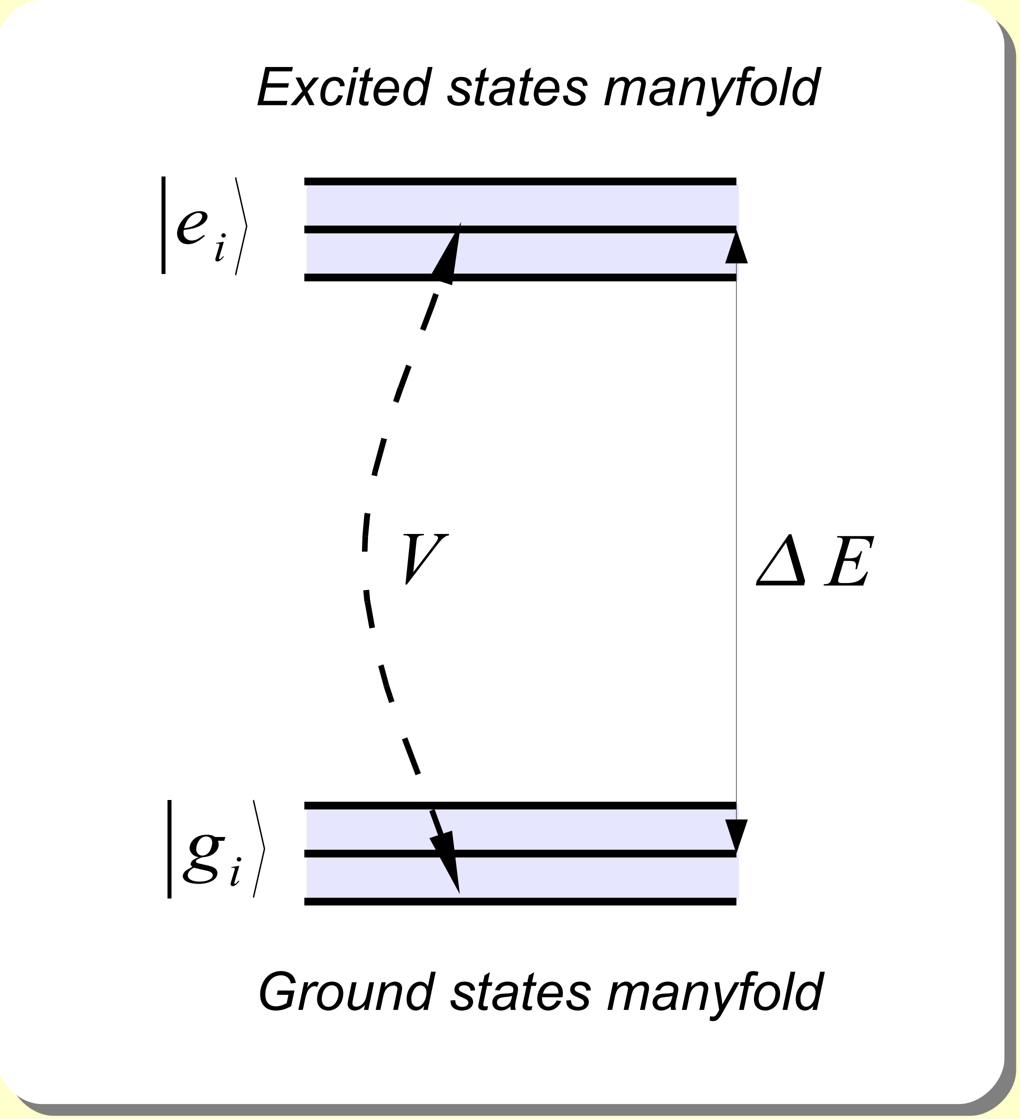
2nd order perturbation theory



- We have two sets of energy levels well separated.
- These sets are coupled by a perturbation

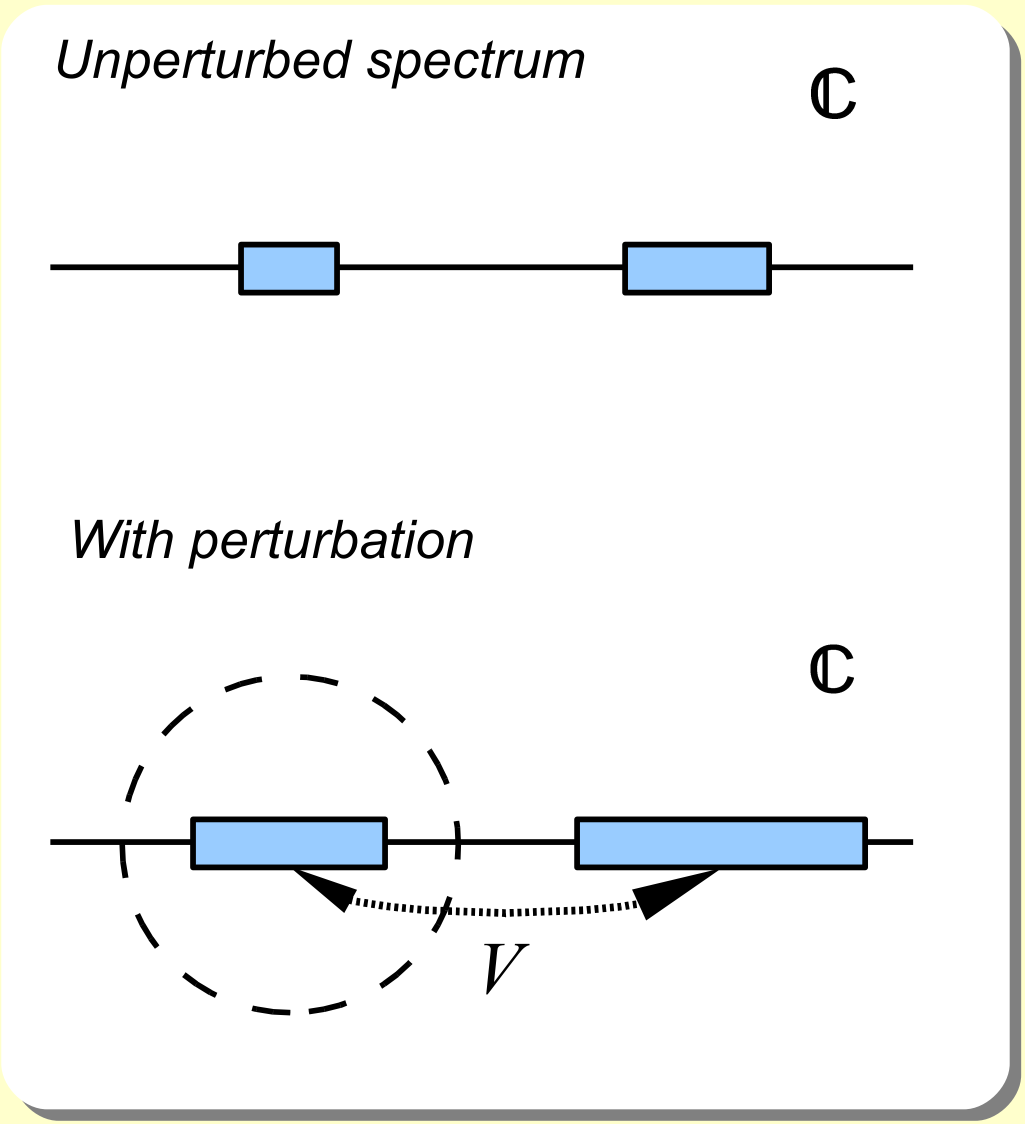
$$V = \sum_{ij} \epsilon_{ij} |e_i\rangle\langle g_j| + H.c.$$

- But the perturbation is weak

$$|\epsilon| \ll |\Delta E|$$

- What happens to the levels?
- Can we produce analytic formulas for E, H, etc?

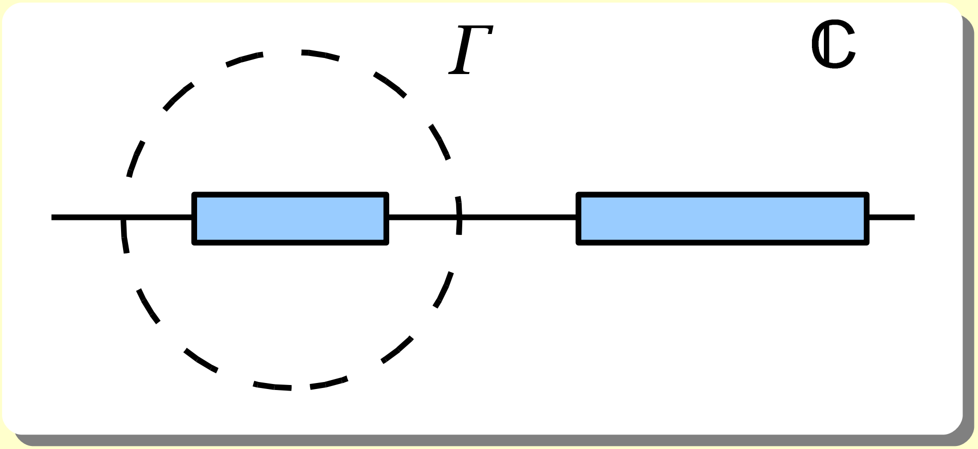
Spectral perturbations



- We draw the eigenenergies as points in the complex plane
  - Several separated interavals.
  - Connected by a small perturbation.
- Theorem: if the perturbation is small, the intervals remain separated.

Spectral perturbations

- We can compute the projector onto the new eigenspace using a trick called “resolvent”



$$P_g = \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{\lambda - H(\epsilon)} d\lambda = \sum_i E_i |g_i\rangle\langle g_i|$$

- This integral gives a sum of projectors onto the individual eigenspaces.

Kato perturbation theory

- We consider both the perturbed and unperturbed problems

$$G_0(z) = \frac{1}{z - H_0} \qquad G_0(z) = \frac{1}{z - H} \qquad (H = H_0 + \epsilon V)$$

- And the perturbed and unperturbed projectors they give rise to

$$P_g^{(0)} = \oint_\Gamma G_0(z)\, dz \qquad P_g = \oint_\Gamma G(z)\, dz$$

- Using the relation $G = G_0 + \epsilon G_0 V G$ we can iteratively build a series for the projectors and for the Hamiltonian

$$P_g = P_g^{(0)} + \sum_n \epsilon^n A^{(n)}$$

$$H P_g = H_0 + \sum_n \epsilon^n B^{(n)}$$

2^{nd} order formulas

- The most relevant formula is the effective Hamiltonian

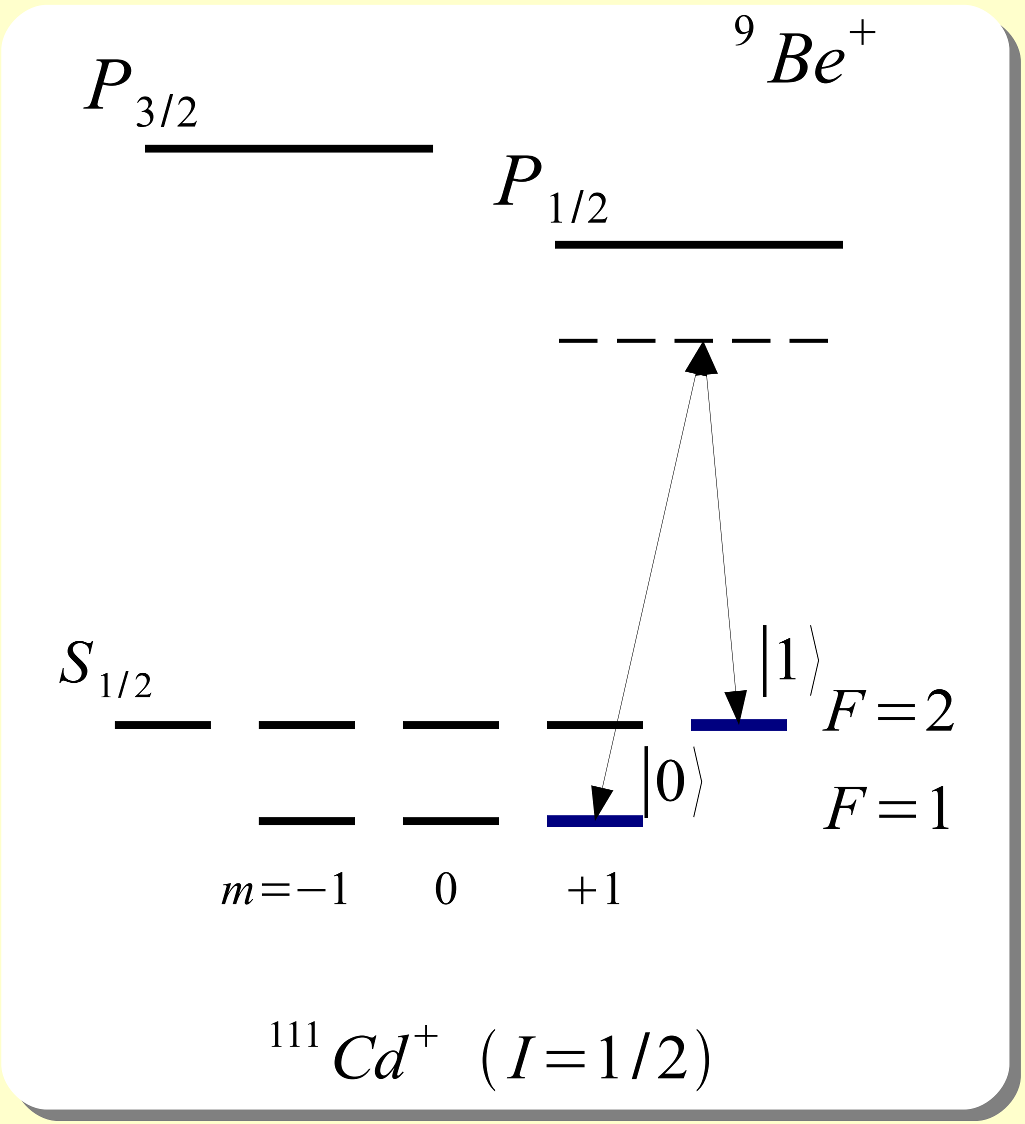
$$H P_g = H_0 + \sum_n \epsilon^n \sum_{k_1+\dots k_n=n} S^{k_1} V S^{k_2} \cdots S^{k_{n-1}} V S^{k_n}$$

$$S^0 \equiv -P^{(0)}, \quad S^{(k>0)} \equiv \left[(E_g - H_0)^{-1} (1 - P^{(0)}) \right]^k$$

- Up to second order we recover the well known formula

$$H_{eff} = H_0 + \epsilon \sum_{g,g'} |g\rangle\langle g|V|g'\rangle\langle g'| +$$
$$-\epsilon^2 \sum_{g,g'} \sum_e |g\rangle\langle g|V|e\rangle\langle e|V|g'\rangle\langle g'| \times$$
$$\times \frac{1}{2} \left[\frac{1}{E_g - E_e} + \frac{1}{E_{g'} - E_e} \right]$$

Raman transitions



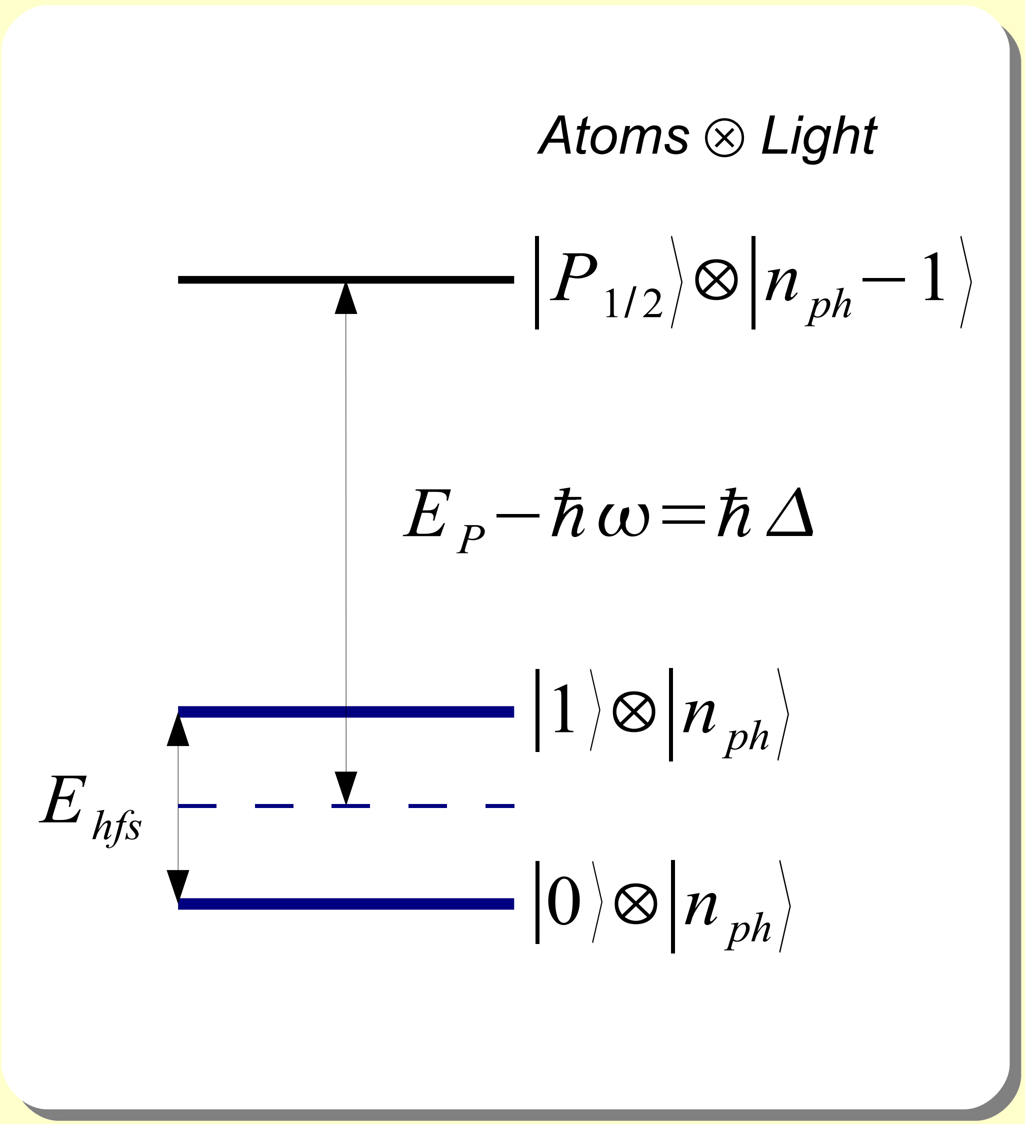
- Unperturbed Hamiltonian for atoms and photons

$$H_0 = E_{hfs}\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right) + E_P |P_{1/2}\rangle\langle P_{1/2}| + \hbar\omega a^{+} a$$

- To this we add a coupling to the excited state

$$H_{pert} = \Omega |P_{1/2}\rangle\langle 0| a + \Omega |P_{1/2}\rangle\langle 1| a + H.c$$

Raman transitions



- Unperturbed Hamiltonian for atoms and photons

$$
\begin{aligned}
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& + \hbar\omega a^+ a
\end{aligned}
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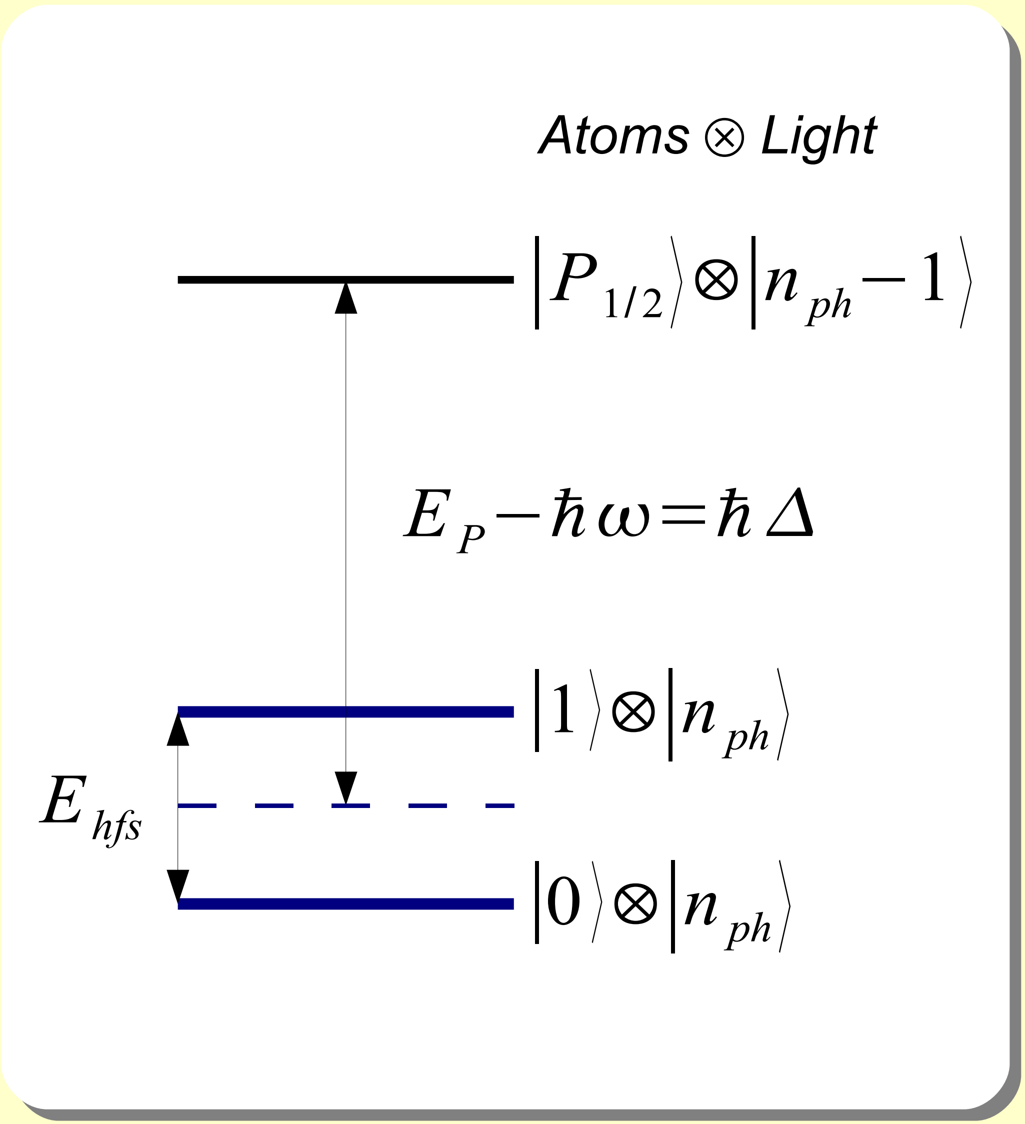
Raman transitions



- Apply the second order formula by Kato

$$H \simeq H_0 - \hbar\Omega^2 n\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)\times$$
$$\times\frac{1}{2}\left[\frac{1}{\Delta - E_{hfs}/2} + \frac{1}{\Delta + E_{hfs}/2}\right]$$

- With good approximation,

$$H_{eff} \sim \hbar\frac{\Omega^2}{\Delta}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)$$
$$+\frac{E_{hfs}}{2}\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right)$$